THE BV-ALGEBRA STRUCTURE OF $\mathcal{W}_3$ COHOMOLOGY [♭]

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Abstract: We summarize some recent results obtained in collaboration with J. McCarthy on the spectrum of physical states in $\mathcal{W}_3$ gravity coupled to $c=2$ matter. We show that the space of physical states, defined as a semi-infinite (or BRST) cohomology of the $\mathcal{W}_3$ algebra, carries the structure of a BV-algebra. This BV-algebra has a quotient which is isomorphic to the BV-algebra of polyvector fields on the base affine space of $SL(3,\mathbb{C})$. Details will appear elsewhere.

1. Introduction

Understanding the spectrum of physical states in theories of two-dimensional $\mathcal{W}$-gravity coupled to matter poses an interesting challenge. Unlike in the case of ordinary gravity, the computation of the relevant semi-infinite (or BRST) cohomology of the underlying $\mathcal{W}$-algebra appears to be very difficult, and only a small number of results have been rigorously established. One expects that by studying the structure of this cohomology space it might be possible to achieve a better understanding of (quantum) $\mathcal{W}$-geometry and string field theory. The problem is also mathematically quite interesting as it involves generalizing some of the standard techniques for computing semi-infinite cohomologies to non-linear algebras.

In this paper we summarize some recent work done in collaboration with J. McCarthy on the computation of physical states in $\mathcal{W}_3$-gravity coupled to two scalar fields, as the semi-infinite cohomology of a tensor product of two Fock space modules of the $\mathcal{W}_3$ algebra. A complete result for the cohomology is given in Conjecture 3.1, Theorem 3.2 and Corollary 3.3. We then discuss in some detail the structure of the space of physical states as a Batalin-Vilkovisky (BV) algebra and, in particular, show that it is modelled on the well-known BV-algebra of regular polyvector fields on the base affine space of $SL(3,\mathbb{C})$. The main result here is given in Theorem 4.6. For more details we refer to [1–3] and the forthcoming paper [4].

Throughout this paper we will use the notation $\mathfrak{h}$ for the Cartan subalgebra, $\mathfrak{h}^*_{\mathbb{Z}}$ for the set of integral weights, P_+ for the set of dominant integral weights, P_{++} for the set of strictly dominant integral weights, Δ_+ for the positive roots and W for the Weyl group of some Lie algebra $\mathfrak{g}$. $\mathcal{L}(\Lambda)$ will denote the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\Lambda \in P_+$ and $\ell(w)$ the length of $w \in W$. In the following $\mathfrak{g}$ will always refer to $\mathfrak{sl}_3$.

2. The $\mathcal{W}_3$ algebra and its modules

The $\mathcal{W}_3$ algebra with central charge $c \in \mathbb{C}$ (denoted simply by $\mathcal{W}$ in the sequel) is defined as the quotient of the free Lie algebra generated by L_m, W_m, $m \in \mathbb{Z}$, by the ideal generated by the following commutation

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relations (see *e.g.* the review on $\mathcal{W}$-algebras [5], and references therein).

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\
[L_m, W_n] &= (2m-n)W_{m+n}\,, \\
[W_m, W_n] &= (m-n)\left(\tfrac{1}{15}(m+n+3)(m+n+2) - \tfrac{1}{6}(m+2)(n+2)\right)L_{m+n} \\
&\quad + \beta(m-n)\Lambda_{m+n} + \tfrac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}\,,
\end{aligned}
\tag{2.1}
$$

where $\beta = 16/(22+5c)$ and

$$
\Lambda_m = \sum_{n\leq -2} L_n L_{m-n} + \sum_{n>-2} L_n L_{m-n} - \tfrac{3}{10}(m+3)(m+2)L_m\,. \tag{2.2}
$$

Notice that, due to the non-linearity of Λ_m in (2.1), $\mathcal{W}$ is *not* a Lie algebra. The Cartan subalgebra $\mathcal{W}_0$ of $\mathcal{W}$ is spanned by L_0 and W_0, but, because $(\operatorname{ad} W_0)$ is not diagonalizable, $\mathcal{W}$ does not admit a root space decomposition (a generalized root space decomposition, *i.e.* a Jordan normal form, does however exist). Nevertheless, it is still convenient to decompose the generators of $\mathcal{W}$ according to the $(-\operatorname{ad} L_0)$ eigenvalue, and define $\mathcal{W}_\pm = \{L_n, W_n \mid \pm n > 0\}$. However, this is not a triangular decomposition in the usual sense.

For physical applications the most interesting representations of $\mathcal{W}$ are the so-called positive energy modules, which are defined by the condition that (the energy operator) L_0 is diagonalizable with finite dimensional eigenspaces, and with the spectrum bounded from below. If the lowest energy eigenspace is one dimensional, we denote the eigenvalues of L_0 and W_0 on the highest weight state by h and w, respectively.

In particular, the Verma module $M(h,w,c)$ is defined as the (positive energy) module induced by $\mathcal{W}_-$ from an 1-dimensional representation of $\mathcal{W}_0$. By the standard argument, $M(h,w,c)$ contains a maximal submodule. We denote the corresponding irreducible quotient module by $L(h,w,c)$. The module contragradient to $M(h,w,c)$ will be denoted by $\overline{M}(h,w,c)$.

Another class of positive energy modules of $\mathcal{W}$ are the Fock space modules $F(\Lambda,\alpha_0)$, which arise in the free field realization of $\mathcal{W}$ in terms of two scalar fields (see *e.g.* [5], and references therein). The modules $F(\Lambda,\alpha_0)$ are labelled by the background charge $\alpha_0 \in \mathbb{C}$ and an $\mathfrak{sl}_3$ weight Λ.

The central charge c and the highest weights h and w of $F(\Lambda,\alpha_0)$ are given by

$$
\begin{aligned}
c(\alpha_0) &= 2 - 24{\alpha_0}^2\,, \\
h(\Lambda) &= -(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) - {\alpha_0}^2 = \tfrac{1}{2}(\Lambda, \Lambda + 2\alpha_0\rho)\,, \\
w(\Lambda) &= \sqrt{3\beta}\,\theta_1\theta_2\theta_3\,,
\end{aligned}
\tag{2.3}
$$

where

$$
\theta_1 = (\Lambda+\alpha_0\rho, \Lambda_1)\,, \quad \theta_2 = (\Lambda+\alpha_0\rho, \Lambda_2-\Lambda_1)\,, \quad \theta_3 = (\Lambda+\alpha_0\rho, -\Lambda_2)\,. \tag{2.4}
$$

Here, Λ_1 and Λ_2 are the fundamental weights of $\mathfrak{sl}_3$, and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the Weyl vector. Note that $h(\Lambda)$ and $w(\Lambda)$ as in (2.3) determine Λ only up to a Weyl rotation $\Lambda \to w(\Lambda+\alpha_0\rho) - \alpha_0\rho$, $w \in W$.

The following theorem summarizes some of the known results on the structure of Fock space modules $F(\Lambda,\alpha_0)$:

Theorem 2.1 [1,2].

(i) Let $\imath'$ and $\imath''$ be the canonical ($\mathcal{W}$-) homomorphisms

$$
M(h(\Lambda), w(\Lambda), c(\alpha_0)) \quad \xrightarrow{\imath'} \quad F(\Lambda,\alpha_0) \quad \xrightarrow{\imath''} \quad \overline{M}(h(\Lambda), w(\Lambda), c(\alpha_0))\,. \tag{2.5}
$$

Then $\imath'$ (resp. $\imath''$) is an isomorphism if $i(\Lambda+\alpha_0\rho) \in \eta D_+$ (resp. $-i(\Lambda+\alpha_0\rho) \in \eta D_+$) and ${\alpha_0}^2 \leq -4$. Here $D_+ = \{\lambda \in \mathfrak{h}^* | (\lambda,\alpha) \geq 0\ \forall\alpha\in\Delta_+\}$ denotes the fundamental Weyl chamber and $\eta \equiv \operatorname{sign}(-i\alpha_0)$.

(ii) For $c=2$, the Fock space $F(\lambda,0)$ is completely reducible. Explicitly, for all $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$, we have

$$F(\lambda,0)\ \cong\ \bigoplus_{\Lambda\in P_+}\ m_\lambda^\Lambda\, L(h(\Lambda),w(\Lambda),2)\,, \tag{2.6}$$

where m_λ^Λ is equal to the multiplicity of the weight λ in the irreducible finite dimensional representation $\mathcal{L}(\Lambda)$ of $\mathfrak{sl}_3$ with highest weight Λ.

3. Fock space cohomology of the $\mathcal{W}_3$ algebra

Despite the fact that $\mathcal{W}$ is not a Lie algebra, the analog of semi-infinite (or BRST-) cohomology can still be defined [6,7]. As usual, one introduces two sets of ghost operators $(b_m^{[j]},c_m^{[j]})\,,j=2,3$ of conformal dimension $(j,-j+1)$, corresponding to the generators L_m and W_m, $m\in\mathbb{Z}$, respectively. These ghost operators satisfy anti-commutation relations $\{b_m^{[j]},c_n^{[j']}\}=\delta_{m+n,0}\delta^{j,j'}$. Let $F^{\rm gh}$ denote the standard positive energy module. The ghost Fock space $F^{\rm gh}=\bigoplus_{n\in\mathbb{Z}}F^{{\rm gh},n}$ is graded by ghost number, where ${\rm gh}(c_m^{[j]})=-{\rm gh}(b_m^{[j]})=1$ and the highest weight state (physical vacuum) is chosen to have ghost number 3 (*i.e.* such that states and their corresponding operators have identical ghost numbers). For any two positive energy modules V^M and V^L, such that $c^M+c^L=100$, there exists a complex $(V^M\otimes V^L\otimes F^{{\rm gh},n},d)$, graded by ghost number, and with a differential (BRST operator) d of degree 1. For an explicit formula for d, which is rather involved, we refer to [7,1,2]. We will denote the cohomology of this complex by $H(\mathcal{W},V^M\otimes V^L)$. The cohomology relative to the Cartan subalgebra $\mathcal{W}_0$ will be denoted by $H(\mathcal{W},\mathcal{W}_0;V^M\otimes V^L)$.

For $V^L\cong F(\Lambda^L,\alpha_0^L)$ this cohomology is interpretated as the set of physical states in $\mathcal{W}$-gravity coupled to some matter theory represented by V^M. One is interested mainly in two cases: where V^M is either a so-called minimal model $L(h^M,w^M,c^M)$ or a free field Fock space $F(\Lambda^M,\alpha_0^M)$. The minimal model case was discussed in [1,3]. The analysis of $H(\mathcal{W},F(\Lambda^M,\alpha_0^M)\otimes F(\Lambda^L,\alpha_0^L))$ for generic α_+ (*i.e.* ${\alpha_+}^2\notin\mathbb{Q}$ where we have parametrized $\alpha_0^M=\alpha_++\alpha_-\,,-i\alpha_0^L=\alpha_+-\alpha_-\,,\alpha_+\alpha_-=-1$) was started in [7] and completed in [3]. Here we will complete the analysis, begun in [2], of a non-generic case, namely $\alpha_\pm=\pm1$ (*i.e.* $\alpha_0^M=0,-i\alpha_0^L=2$ or $c^M=2,c^L=98$).

Because of Theorem 2.1 (ii) it suffices to compute the cohomology for the $c=2$ irreducible $\mathcal{W}$-modules $L(\Lambda)\equiv L(h(\Lambda),w(\Lambda),2)$

Conjecture 3.1 [4]. *Let $\Lambda\in P_+$.*

(i) The cohomology $H^n(\mathcal{W},\mathcal{W}_0;L(\Lambda)\otimes F(\Lambda^L,2i))$ is nontrivial only if there exist $w\in W,\ \sigma\in W\cup\{0\}$ such that

$$-i\Lambda^L+2\rho=w^{-1}(\Lambda+\rho-\sigma\rho)\,. \tag{3.1}$$

(ii) For w,σ,Λ and Λ^L as in (3.1), the cohomology $H^n(\mathcal{W},\mathcal{W}_0;L(\Lambda)\otimes F(\Lambda^L,2i))$ is 1-dimensional in the following cases

$$\begin{array}{cccc}
\sigma\in W\,, & \Lambda\in P_+\,, & w\in W\,, & n=\ell(w^{-1})-\ell(w^{-1}\sigma)+3\,,\\
\sigma=0\,, & \Lambda\in P_{++}\,, & w\in W\,, & n=\ell(w^{-1})+1 \text{ or } n=\ell(w^{-1})+2\,,\\
\sigma=0\,, & (\Lambda,\alpha_i)=0\,,\Lambda\neq0\,, & w\in\,\langle r_i\rangle\backslash W\,, & n=\ell(w^{-1})+2\,,\\
\sigma=0\,, & (\Lambda,\alpha_i)=0\,,\Lambda\neq0\,, & w\in r_i(\langle r_i\rangle\backslash W)\,, & n=\ell(w^{-1})+1\,.
\end{array}$$

and vanishes otherwise.

In the case that certain weights $(\Lambda,-i\Lambda^L)$ and certain ghost number n satisfy (i) and (ii) for more than one choice of (w,σ), the above should be understood in the sense that the corresponding cohomology is nevertheless 1-dimensional.

Let us comment on the status of this conjecture. For $-i\Lambda^L + 2\rho \in P_+$ we have an isomorphism $F(\Lambda^L, 2i) \cong \overline{M}(h(\Lambda^L), w(\Lambda^L), 2)$ (see Theorem 2.1 *(i)*). By taking the (conjectured) resolutions of $L(\Lambda)$ in terms of generalized Verma modules $M(h, w, c = 2)_N$ [2] and using the known result for $H^n(\mathcal{W}, \mathcal{W}_0; M(h, w, c)\otimes \overline{M}(h', w', 100 - c))$, the conjecture follows (see [2] for details). [The resolution of $L(\Lambda)$ for $\Lambda \in P_{++}$ in [2] contains a minor misprint, see [4].]

For the other Weyl chambers, *i.e.* $w(-i\Lambda^L + 2\rho) \in P_+$, the conjecture is based on an analysis of the cohomology for generic α_+ in the limit $\alpha_+ \to 1$ (*i.e.* $c^M \to 2$) and passes various nontrivial consistency checks. Among others, it is consistent with duality

$$H^{6-n}(\mathcal{W}, \mathcal{W}_0; L(\Lambda) \otimes F(\Lambda^L, 2i)) \cong H^n(\mathcal{W}, \mathcal{W}_0; L(\Lambda) \otimes \overline{F}(\Lambda^L, 2i)) , \tag{3.2}$$

where $\overline{F}(\Lambda, \alpha_0) \cong F(w_0(\Lambda + \alpha_0\rho) - \alpha_0\rho)$ denotes the module contragradient to $F(\Lambda, \alpha_0)$.

Both the conjectured resolutions of $L(\Lambda)$ as well as the result for the cohomology (Conjecture 3.1) have also been verified by extensive computer calculations using Mathematica™.

Let L be the lattice

$$L \;\equiv\; \{(\lambda, \mu) \in \mathfrak{h}^*_{\mathbb{Z}} \otimes \mathfrak{h}^*_{\mathbb{Z}} \,|\, \lambda - \mu \in \mathbb{Z} \cdot \Delta_+\} . \tag{3.3}$$

Note that, in particular,

$$(\lambda, \lambda') - (\mu, \mu') = (\lambda - \mu, \lambda') + (\mu, \lambda' - \mu') \in \mathbb{Z} , \tag{3.4}$$

for all pairs (λ, μ) and (λ', μ') in L. We will restrict the momenta $(\Lambda^M, -i\Lambda^L)$ to the lattice L. As a consequence, all the vertex operators $V_{(\Lambda^M,\Lambda^L)}(z) = \exp(i\Lambda^M \cdot \phi^M + i\Lambda^L \cdot \phi^L)(z)$ will become mutually local because of (3.4) and, moreover, one can find a set of cocycles turning the underlying BRST-complex into a Vertex Operator Algebra (VOA). This will be essential for the construction of the BV-algebra in Section 3. In addition, the most interesting cohomology happens to be situated at $(\Lambda^M, -i\Lambda^L) \in L$.

Now consider the cohomologies

$$\begin{aligned} \mathcal{H} &= \bigoplus_{(\Lambda^M,-i\Lambda^L)\in L} H(\mathcal{W}, F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)) , \\ \mathcal{H}_{\rm rel} &= \bigoplus_{(\Lambda^M,-i\Lambda^L)\in L} H(\mathcal{W}, \mathcal{W}_0; F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)) . \end{aligned} \tag{3.5}$$

We recall

Theorem 3.2 [1,2].

(i) *$\mathcal{H}$ (and $\mathcal{H}_{\rm rel}$) carries the structure of a $\mathfrak{g}\oplus\mathfrak{h}$ module ($\mathfrak{g} \cong \mathfrak{sl}_3$). The action of $\mathfrak{g}$ is through the zero modes of the Frenkel-Kac-Segal vertex operator construction (in matter fields only), while $\mathfrak{h}$ acts as $-ip^L$ (with eigenvalues $-i\Lambda^L$). This module is completely reducible under $\mathfrak{g} \oplus \mathfrak{h}$.*

(ii) *There exists a (non-canonical) isomorphism (as $\mathfrak{g} \oplus \mathfrak{h}$ modules)*

$$\mathcal{H}^i \;\cong\; \mathcal{H}^i_{\rm rel} \oplus \mathcal{H}^{i-1}_{\rm rel} \oplus \mathcal{H}^{i-1}_{\rm rel} \oplus \mathcal{H}^{i-2}_{\rm rel} .$$

By combining the results of Theorems 2.1, 3.2 and Conjecture 3.1, we find

Corollary 3.3. *The cohomology $\mathcal{H}_{\rm rel}$ is isomorphic (as a $\mathfrak{g} \oplus \mathfrak{h}$ module) to the direct sum of irreducible modules $\mathcal{L}(\Lambda)\otimes\mathbb{C}_{\Lambda'}$ with momenta $(\Lambda, \Lambda') \in \mathfrak{h}^*_{\mathbb{Z}}\otimes\mathfrak{h}^*_{\mathbb{Z}}$ lying in a set of disjoint cones $\{\mathcal{S}^n_w + (\lambda, w^{-1}\lambda) \,|\, \lambda \in P_+\}$, i.e.*

$$\mathcal{H}^n_{\rm rel} \;\cong\; \bigoplus_{w\in W} \; \bigoplus_{(\Lambda,\Lambda')\in\mathcal{S}^n_w} \; \bigoplus_{\lambda\in P_+} \left(\mathcal{L}(\Lambda + \lambda) \otimes \mathbb{C}_{\Lambda'+w^{-1}\lambda}\right) ,$$

where the sets $\mathcal{S}^n_w$ (tips of the cones) are given in Table 1.

n	w	$\mathcal{S}_w^n$
0	1	$(0,0)$
1	1	$(\Lambda_1,-\Lambda_1+\Lambda_2)$, $(\Lambda_1+\Lambda_2,0)$, $(\Lambda_2,\Lambda_1-\Lambda_2)$
	r_1	$(0,-2\Lambda_1+\Lambda_2)$
	r_2	$(0,\Lambda_1-2\Lambda_2)$
2	1	$(2\Lambda_1,-\Lambda_1)$, $(0,-\Lambda_1-\Lambda_2)$, $(2\Lambda_2,-\Lambda_2)$
	r_1	$(\Lambda_1,-2\Lambda_1)$, $(\Lambda_2,-3\Lambda_1+\Lambda_2)$, $(0,-4\Lambda_1+2\Lambda_2)$
	r_2	$(\Lambda_2,-2\Lambda_2)$, $(\Lambda_1,\Lambda_1-3\Lambda_2)$, $(0,2\Lambda_1-4\Lambda_2)$
	r_2r_1	$(0,-3\Lambda_1)$
	r_1r_2	$(0,-3\Lambda_2)$
3	1	$(\Lambda_1+\Lambda_2,-\Lambda_1-\Lambda_2)$
	r_1	$(\Lambda_2,-2\Lambda_1-\Lambda_2)$, $(\Lambda_1,-4\Lambda_1+\Lambda_2)$, $(\Lambda_2,-5\Lambda_1+2\Lambda_2)$
	r_2	$(\Lambda_1,-\Lambda_1-2\Lambda_2)$, $(\Lambda_2,\Lambda_1-4\Lambda_2)$, $(\Lambda_1,2\Lambda_1-5\Lambda_2)$
	r_2r_1	$(\Lambda_1,-3\Lambda_1-\Lambda_2)$, $(0,-5\Lambda_1+\Lambda_2)$, $(\Lambda_1,-5\Lambda_1)$
	r_1r_2	$(\Lambda_2,-\Lambda_1-3\Lambda_2)$, $(0,\Lambda_1-5\Lambda_2)$, $(\Lambda_2,-5\Lambda_2)$
	$r_1r_2r_1$	$(0,-2\Lambda_1-2\Lambda_2)$
4	r_1	$(0,-4\Lambda_1-\Lambda_2)$
	r_2	$(0,-\Lambda_1-4\Lambda_2)$
	r_2r_1	$(\Lambda_1,-4\Lambda_1-2\Lambda_2)$, $(\Lambda_2,-5\Lambda_1-\Lambda_2)$, $(0,-6\Lambda_1)$
	r_1r_2	$(\Lambda_2,-2\Lambda_1-4\Lambda_2)$, $(\Lambda_1,-\Lambda_1-5\Lambda_2)$, $(0,-6\Lambda_2)$
	$r_1r_2r_1$	$(0,-3\Lambda_1-3\Lambda_2)$, $(2\Lambda_1,-4\Lambda_1-3\Lambda_2)$, $(2\Lambda_2,-3\Lambda_1-4\Lambda_2)$
5	r_2r_1	$(0,-5\Lambda_1-2\Lambda_2)$
	r_1r_2	$(0,-2\Lambda_1-5\Lambda_2)$
	$r_1r_2r_1$	$(\Lambda_1,-5\Lambda_1-3\Lambda_2)$, $(\Lambda_1+\Lambda_2,-4\Lambda_1-4\Lambda_2)$, $(\Lambda_2,-3\Lambda_1-5\Lambda_2)$
6	$r_1r_2r_1$	$(0,-4\Lambda_1-4\Lambda_2)$

Table 1. The sets $\mathcal{S}_w^n$

In particular we see that, as an $\mathfrak{sl}_3$ module, the 'ground ring' $\mathcal{H}^0$ decomposes as $\mathcal{H}^0 \cong \bigoplus_{\Lambda\in P_+} \mathcal{L}(\Lambda)$ and is therefore a so-called 'model space' for $\mathfrak{sl}_3$. It is well-known that this model space can be realized as the space $\mathcal{P}^0(A)$ of polynomial functions on the so-called 'base-affine space' $A \equiv N_+\backslash G$ [8]. For $\mathfrak{sl}_3$ this model space is given by $\mathbb{C}\,[x^i, y_i]/\langle x^i y_i\rangle$ $(i = 1,2,3)$, *i.e.* polynomials in 6 variables x^i, y_i transforming in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $\mathfrak{sl}_3$ respectively, with a single relation $x^i y_i = 0$ [9]. In fact, one can show that $\mathcal{H}^0 \cong \mathcal{P}^0(A)$ as algebras [4]. One might think that, just as in the Virasoro case (corresponding to $\mathfrak{g} \cong \mathfrak{sl}_2$) [10–12], part of the rest of $\mathcal{H}$ allow an interpretation in terms of polyvector fields on this base affine space. This turns out to be true and will be elaborated on in the next section.

4. The BV-structure of $\mathcal{H}$

To explain the algebraic structure of the cohomology $\mathcal{H}$ of Section 2 we will first need to recall the definition of a Gerstenhaber algebra (or G-algebra, for short) [13] and a BV-algebra (or coboundary G-algebra) [14–16,12] as well as some basic facts.

Definition 4.1. *A G-algebra $(\mathcal{A}, \cdot, [\ ,\ ])$ is a $\mathbb{Z}$-graded, supercommutative, associative algebra $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i$ (under $\cdot$) as well as a $\mathbb{Z}$-graded Lie superalgebra (under $[\ ,\ ]$), such that the (odd) bracket acts as a superderivation of the algebra, i.e.*

$$[x, y\cdot z] = [x,y]\cdot z + (-1)^{(|x|-1)|y|} y\cdot [x,z]\,, \qquad x,y,z\in\mathcal{A}\,. \tag{4.1}$$

For any commutative algebra $\mathcal{A}$ and $\mathcal{A}$-module $\mathcal{M}$, one defines the the set $\mathcal{D}(\mathcal{A},\mathcal{M})$ of derivations of $\mathcal{A}$ with coefficients in $\mathcal{M}$ as the set of elements $D\in \mathrm{Hom}(\mathcal{A},\mathcal{M})$ that satisfy the Leibniz rule

$$D(x\cdot y) = y(Dx) + x(Dy)\,. \tag{4.2}$$

The set $\mathcal{D}^n(\mathcal{A})$ of polyderivations of order n is defined by induction as those $D\in\mathrm{Hom}(\mathcal{A},\mathcal{D}^{n-1}(\mathcal{A}))$ satisfying the Leibniz rule (4.2) as well as being completely antisymmetric when considered as elements of $\mathrm{Hom}(\mathcal{A}^{\otimes n},\mathcal{A})$. We recall

Theorem 4.2 [17]. *Let $\mathcal{A}$ be a commutative algebra. The set of polyderivations $\mathcal{D}(\mathcal{A})$ carries the structure of a G-algebra, with the bracket given by the Schouten bracket.*

Another example of a G-algebra is the Hochschild cohomology $H(\mathcal{A},\mathcal{A})$ of an associative algebra $\mathcal{A}$ [13].

Definition 4.3. *A BV-algebra $(\mathcal{A}, \cdot, \Delta)$ is a $\mathbb{Z}$-graded, supercommutative, associative algebra $\mathcal{A}$ with a second order derivation Δ (BV-operator) of degree -1 satisfying $\Delta^2 = 0$.*

Lemma 4.4 [18,12,16]. *For any BV-algebra $(\mathcal{A}, \cdot, \Delta)$ we may define an odd bracket by*

$$[x,y] = (-1)^{|x|}\Big(\Delta(x\cdot y) - (\Delta x)\cdot y - (-1)^{|x|} x\cdot(\Delta y)\Big)\,, \qquad x,y\in\mathcal{A}\,. \tag{4.3}$$

This will equip $\mathcal{A}$ with the structure of a G-algebra. Moreover, the BV-operator acts as a superderivation of the bracket

$$\Delta[x,y] = [\Delta x, y] + (-1)^{|x|-1}[x,\Delta y]\,. \tag{4.4}$$

In general, given a commutative algebra $\mathcal{A}$, the G-algebra $\mathcal{D}(\mathcal{A})$ of polyderivations of $\mathcal{A}$ will not carry the structure of a BV-algebra. However, if $\mathcal{A}$ is the algebra of (smooth or polynomial) functions on some smooth manifold M, then $\mathcal{D}(\mathcal{A})$ is isomorphic to the set of polyvector fields $\mathcal{P}(M)$ on M [17]. If, moreover, M possesses a volume form, then we can in fact equip $\mathcal{D}(\mathcal{A})$ $(=\mathcal{P}(M))$ with the structure of a BV-algebra [18,12]. Another example of a BV-algebra is the Grassmann algebra $\bigwedge^* \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ [12].

Given a BV-algebra $(\mathcal{A}, \cdot, \Delta)$, let $\mathcal{A}^0$ be its 'ground ring.' It follows from equations (4.1), (4.3) and (4.4) that there exists a natural way to embed $\mathcal{A}$ into the G-algebra of polyderivations of $\mathcal{A}^0$, *i.e.* $\mathcal{D}(\mathcal{A}^0)$, namely

Theorem 4.5. *Let $(\mathcal{A}, \cdot, \Delta)$ be a BV-algebra. Suppose $\mathcal{A}^n = 0$ for all $n<0$.*

(i) There exists a homomorphism of G-algebras $\pi : \mathcal{A}\to\mathcal{D}(\mathcal{A}^0)$ defined by

$$\pi(y)(x_1,x_2,\ldots,x_n) = [[\ldots[[y,x_1],x_2],\ldots],x_n]\,, \qquad y\in\mathcal{A}^n,\ x_1,x_2,\ldots,x_n\in\mathcal{A}^0\,. \tag{4.5}$$

(ii) Suppose that the G-algebra $\mathcal{D}(\mathcal{A}^0)$ admits a BV-structure $(\mathcal{D}(\mathcal{A}^0), \cdot, \Delta')$ and that $\pi\Delta(x) = \Delta'\pi(x)$ for all $x\in\mathcal{A}^1$, then π is a BV-homomorphism and $\mathcal{I}\equiv\mathrm{Ker}\,\pi$ is a BV-ideal of $\mathcal{A}$.

We are now ready to state the main result of this paper

THEOREM 4.6. *Let $\mathcal{H}$ be the cohomology defined in (3.5). Then*
(i) $\mathcal{H}$ can be equipped with the structure of a BV-algebra.
(ii) There exists an ideal $\mathcal{I} \subset \mathcal{H}$ such that we have an exact sequence of BV-algebras

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{H} \xrightarrow{\pi} \mathcal{D}(\mathcal{H}^0) \longrightarrow 0\,, \tag{4.6}$$

where $\mathcal{D}(\mathcal{H}^0)$ is isomorphic to the BV-algebra $\mathcal{P}(A)$ of polyvector fields on the base affine space $A = N_+\backslash G$.

Let us make some comments on the proof. Quite generally, as has been shown in [10–12,16], BRST cohomologies of VOA's carry the structure of a BV-algebra. The product in this BV-algebra is given by the normal ordered product of the VOA while $\Delta = b_0^{[2]}$. The crucial part of the proof of *(i)* is therefore to show that the complex carries the structure of a VOA. This amounts to showing that one can find an appropriate set of cocycles for the lattice L. This is a straightforward exercise. [One might wonder whether there exists additional structure in $\mathcal{H}$ beyond that of a BV-algebra, in particular whether $b_0^{[3]}$ gives rise to a second BV-operator. It turns out however that, due to the non-diagonalizability of W_0, $b_0^{[3]}$ does *not* act on $\mathcal{H}$.] As we have seen in Section 2, there exists a canonical isomorphism of algebras $\mathcal{H}^0 \cong \mathcal{P}^0(A)$, where $\mathcal{P}^0(A)$ denotes the (commutative) algebra of polynomials on A. This implies $\mathcal{D}(\mathcal{H}^0) \cong \mathcal{P}(A)$ as algebras. That π is in fact a BV-epimorphism follows from Theorem 4.5 by explicitly checking that π intertwines the BV-operators on $\mathcal{H}^1$ and $\mathcal{P}^1(A)$ and that it acts onto.

We would like to remark here that, contrary to the Virasoro case [12], both the dot product and the bracket in $\mathcal{I}$ are not identically zero. Also, the exact sequence (4.6) splits both as an exact sequence of $\mathcal{H}^0$ and $\mathfrak{g} \oplus \mathfrak{h}$ modules, but *not* as an exact sequence of BV-algebras.

Details of this paper as well as a more detailed analysis of the BV-algebra structure of the entire $\mathcal{H}$ will appear elsewhere [4].

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